Appendix 1
A review of some fundamental mathematical and statistical concepts

A1 Introduction

This appendix presents a very brief summary of several important mathematical and statistical concepts. These concepts are, in the opinion of this author, fundamental to a solid understanding of the material of this book. They are presented in an appendix since it is anticipated that the majority of readers will already have some exposure to the techniques, but may require some brief revision. The topics that will be covered are: characteristics of probability distributions and sampling, differential calculus, properties of logarithms and matrix algebra.

A2 Characteristics of probability distributions

A random variable is one that can take on *any value from a given set*. The most commonly used distribution to characterise a random variable is a normal or Gaussian (these terms are equivalent) distribution. The normal distribution is particularly useful since it is symmetric, and the only pieces of information required to completely specify the distribution are its mean and variance.

The probability density function for a normal random variable with mean μ and variance σ^2 is given by $f(y)$ in the following expression[1]

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2}$$

Entering values of y into this expression would trace out the familiar 'bell-shape' of the normal distribution described in chapter 2.

The mean of a random variable y is also known as its expected value, written $\mathrm{E}(y)$. The properties of expected values are used widely in econometrics, and are listed below, referring to a random variable y:

- The expected value of a constant (or a variable that is non-stochastic) is the constant (or non-stochastic variable), e.g. $\mathrm{E}(c) = c$.
- The expected value of a constant multiplied by a random variable is equal to the constant multiplied by the expected value of the variable:

[1] Note that here, we are referring to the density of a single observation for y rather than the joint density of all of the observations.

$E(c\,y) = cE(y)$. It can also be stated that $E(c\,y + d) = (c\,E(y)) + d$, where d is also a constant.
- For two independent random variables, y_1 and y_2, $E(y_1 y_2) = E(y_1)\,E(y_2)$.

The variance of a random variable y is usually written var (y). The properties of the 'variance operator', var, are listed below:

- The variance of a random variable y is given by var $(y) = E[y - E(y)]^2$
- The variance of a constant is zero: var $(c) = 0$
- For c and d constants, var $(c\,y + d) = c^2$ var (y)
- For two independent random variables, y_1 and y_2, var $(c\,y_1 + d y_2) = c^2$ var $(y_1) + d^2$ var (y_2).

The covariance between two random variables, y_1 and y_2, measures the degree of association between them, and is expressed cov (y_1, y_2). The properties of the covariance operator are:

- cov $(y_1, y_2) = E[(y_1 - E(y_1))(y_2 - E(y_2))]$
- For two independent random variables, y_1 and y_2, cov $(y_1, y_2) = 0$
- For four constants, c, d, e, and f, cov $(c + d y_1, e + f y_2) = df$ cov (y_1, y_2).

If a random sample of size T: $y_1, y_2, y_3, \ldots, y_n$ is drawn from a population that is normally distributed with mean μ and variance σ^2, the sample mean, $\bar{y}$ is also normally distributed with mean μ, and variance σ^2/T. In fact, the central limit theorem states that the sampling distribution of the mean of any random sample of observations will tend towards the normal distribution with mean equal to the population mean, μ as the sample size tends to infinity.

A3 Properties of logarithms

Logarithms were invented to simplify cumbersome calculations, since exponents can then be added or subtracted, which is easier than multiplying or dividing the original numbers. While making logarithmic transformations for computational ease is no longer necessary, they still have important uses in algebra and in data analysis. For the latter, there are at least three reasons why log transforms may be useful. First, taking a logarithm can often help to rescale the data so that their variance is more constant, which overcomes a common statistical problem. Second, logarithmic transforms can help to make a positively skewed distribution closer to a normal distribution. Third, taking logarithms can also be a way to make a non-linear, multiplicative relationship between variables into a linear, additive one. These issues are discussed in some detail in chapter 4.

Taking a logarithm is the inverse of a taking an exponential. Natural logarithms, also known as logs to base e (where e is 2.71828...), are more commonly used and more useful mathematically than logs to any other bases. A log to base e is known as a natural or Naperian logarithm, denoted interchangeably by $\ln(y)$ or $\log(y)$.

The properties of logarithms or 'laws of logs' are:

- $\ln(x\,y) = \ln(x) + \ln(y)$
- $\ln(x/y) = \ln(x) - \ln(y)$

- $\ln(y^c) = c \ln(y)$
- $\ln(1) = 0$
- $\ln(1/y) = \ln(1) - \ln(y) = -\ln(y)$.

A4 Differential calculus

The effect of the *rate of change of one variable on the rate of change of another* is measured by a mathematical derivative. If the relationship between the two variables can be represented by a curve, the gradient of the curve will be this rate of change. Consider a variable y that is some function f of another variable x, i.e. $y = f(x)$. The derivative of y with respect to x is written

$$\frac{dy}{dx}$$

or sometimes $f'(x)$. This term

$$\frac{dy}{dx}$$

measures the instantaneous rate of change of y with respect to x.

The basic rules of differentiation are as follows:

- The derivative of a constant is zero

  e.g. if $y = 10$, $\dfrac{dy}{dx} = 0$

  This is because $y = 10$ would be represented as a horizontal straight line on a graph of y against x, and therefore the gradient of this function is zero.
- The derivative of a linear function is simply its slope

  e.g. if $y = 3x + 2$, $\dfrac{dy}{dx} = 3$
- The derivative of a power function n of x

  i.e. $y = cx^n$ is given by $\dfrac{dy}{dx} = cnx^{n-1}$

  For example

  $$y = 4x^3,\ \frac{dy}{dx} = (4 \times 3)x^2 = 12x^2$$
  $$y = 3x^{-1},\ \frac{dy}{dx} = (3 \times -1)x^{-2} = -3x^{-2}$$
- The derivative of a sum is equal to the sum of the derivatives of the individual parts. Similarly, the derivative of a difference is equal to the difference of the derivatives of the individual parts

  e.g. if $y = f(x) + g(x)$, $\dfrac{dy}{dx} = f'(x) + g'(x)$

  while

  if $y = f(x) - g(x)$, $\dfrac{dy}{dx} = f'(x) - g'(x)$

- The derivative of the log of x is given by $1/x$

$$\text{i.e. } \frac{d(\log(x))}{dx} = \frac{1}{x}$$

- The derivative of the log of a function is the derivative of the function divided by the function

$$\text{i.e. } \frac{d(\log(f(x)))}{dx} = \frac{f'(x)}{f(x)}$$

For example, the derivative of $\log(x^3 + 2x - 1)$ is given by

$$\frac{3x^2 + 2}{x^3 + 2x - 1}$$

- The derivative of e^x is e^x. The derivative of $e^{f(x)}$ is given by $f'(x)e^{f(x)}$.
- In the case where y is a function of more than one variable (e.g. $y = f(x_1, x_2, \ldots, x_n)$), it may be of interest to determine the effect that changes in each of the individual x variables would have on y. The differentiation of y with respect to only one of the variables, holding the others constant, is known as *partial differentiation*. The partial derivative of y with respect to a variable x_1 is usually denoted

$$\frac{\partial y}{\partial x_1}$$

All of the rules for differentiation explained above still apply. To give an illustration, suppose $y = 3x_1^3 + 4x_1 - 2x_2^4 + 2x_2^2$. The partial derivative of y with respect to x_1 would be

$$\frac{\partial y}{\partial x_1} = 9x_1^2 + 4$$

while the partial derivative of y with respect to x_2 would be

$$\frac{\partial y}{\partial x_2} = -8x_2^3 + 4x_2$$

- The maximum or minimum of a function with respect to a given variable can be found by taking the derivative of the function with respect to that variable and setting it to zero. The reason that the derivative is set to zero is that at a function maximum or minimum, the gradient of the function will be zero. For example, in chapter 3, the OLS estimator gives formulae for the values of the parameters that minimise the residual sums of squares, given by $L = \sum_t (y_t - \hat{\alpha} - \hat{\beta}x_t)^2$. The minimum of L (the residual sum of squares) is found by partially differentiating this function with respect to $\hat{\alpha}$ and $\hat{\beta}$ and setting these partial derivatives to zero.[2]

[2] In fact, we cannot be sure whether the values of $\hat{\alpha}$ and $\hat{\beta}$ found would provide a minimum or a maximum of the residual sum of squares, as both a minimum and a maximum would have first derivatives equal to zero. To determine this would require the calculation of second derivatives of the functions with respect to $\hat{\alpha}$ and $\hat{\beta}$. Second derivatives are not covered in this book, although in the case of the OLS estimator, the values of $\hat{\alpha}$ and $\hat{\beta}$ selected do in fact minimise the residual sums of squares.

A5 Matrices

A matrix is simply a *collection or array of numbers*. The size of a matrix is given by its number of rows and columns. Matrices are very useful and important ways for organising sets of data together, which make manipulating and transforming them much easier than it would be to work with each constituent of the matrix separately. Matrices are widely used in econometrics and in financial theory for deriving key results and for expressing formulae in a succinct way. Some useful features of matrices and explanations of how to work with them are described below:

- The size of a matrix is quoted as $R \times C$, which is the number of rows by the number of columns.
- Each element in a matrix is referred to using subscripts. For example, suppose a matrix M has two rows and four columns. The element in the second row and the third column of this matrix would be denoted m_{23}, so that m_{ij} refers to the element in the ith row and the jth column.
- If a matrix has only one row, it is known as a row vector, which will be of dimension $1 \times C$ e.g. $(2.7 \quad 3.0 \quad -1.5 \quad 0.3)$
- A matrix having only one column is known as a column vector, which will be of dimension $R \times 1$

$$\text{e.g.} \begin{pmatrix} 1.3 \\ -0.1 \\ 0.0 \end{pmatrix}$$

- When the number of rows and columns is equal (i.e. $R = C$), it would be said that the matrix is square

$$\text{e.g.} \begin{pmatrix} 0.3 & 0.6 \\ -0.1 & 0.7 \end{pmatrix}$$

- A matrix in which all the elements are zero is known as a zero matrix

$$\text{e.g.} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- A symmetric matrix is a special type of square matrix that is symmetric about the leading diagonal (the diagonal line running through the matrix from the top left to the bottom right), so that $m_{ij} = m_{ji} \; \forall \, i, j$

$$\text{e.g.} \begin{pmatrix} 1 & 2 & 4 & 7 \\ 2 & -3 & 6 & 9 \\ 4 & 6 & 2 & -8 \\ 7 & 9 & -8 & 0 \end{pmatrix}$$

- A diagonal matrix is a square matrix which has non-zero terms on the leading diagonal and zeros everywhere else

$$\text{e.g.} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- A diagonal matrix with 1 in all places on the leading diagonal and zero everywhere else is known as the identity matrix, denoted by I. By definition, an identity matrix must be symmetric (and therefore also square)

$$\text{e.g.}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The identity matrix is essentially the matrix equivalent of the number one. Multiplying any matrix by the identity matrix of the appropriate size results in the original matrix being left unchanged

$$\text{e.g. } MI = IM = M$$

- In order to perform operations with matrices (e.g. addition, subtraction, or multiplication), the matrices concerned must be *conformable*. The dimensions of matrices required for them to be conformable depend on the operation.
- Addition and subtraction of matrices requires the matrices concerned to be of the same order (i.e. to have the same number of rows and the same number of columns as one another). The operations are then performed element by element.

$$\text{E.g., if } A = \begin{pmatrix} 0.3 & 0.6 \\ -0.1 & 0.7 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0.2 & -0.1 \\ 0 & 0.3 \end{pmatrix},$$

$$A + B = \begin{pmatrix} 0.5 & 0.5 \\ -0.1 & 1.0 \end{pmatrix}, \quad A - B = \begin{pmatrix} 0.1 & 0.7 \\ -0.1 & 0.4 \end{pmatrix}$$

- Multiplying or dividing a matrix by a scalar (that is, a single number), implies that every element of the matrix is multiplied by that number

$$\text{e.g. } 2A = 2\begin{pmatrix} 0.3 & 0.6 \\ -0.1 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.6 & 1.2 \\ -0.2 & 1.4 \end{pmatrix}$$

- It can also be stated that, for two matrices A and B of the same order and for c a scalar

$$A + B = B + A$$

$$A + 0 = 0 + A = A$$

$$cA = A\ c$$

$$c(A + B) = cA + cB$$

$$A0 = 0A = 0$$

- Multiplying two matrices together requires the number of columns of the first matrix to be equal to the number of rows of the second matrix. Note also that the ordering of the matrices is important, so that in general, $AB \neq BA$. When the matrices are multiplied together, the resulting matrix will be of size (number of rows of first matrix $\times$ number of columns of second matrix), e.g. $(3 \times 2) \times (2 \times 4) = (3 \times 4)$. It is as if the columns of the first matrix and the rows of the second cancel out. This rule also follows more generally, so that $(a \times b) \times (b \times c) \times (c \times d) \times (d \times e) = (a \times e)$, etc.

- The actual multiplication of the elements of the two matrices is done by multiplying along the rows of the first matrix and down the columns of the second

$$\text{e.g.} \underset{(3\times 2)}{\begin{pmatrix} 1 & 2 \\ 7 & 3 \\ 1 & 6 \end{pmatrix}} \underset{(2\times 4)}{\begin{pmatrix} 0 & 2 & 4 & 9 \\ 6 & 3 & 0 & 2 \end{pmatrix}}$$

$$= \underset{(3\times 4)}{\begin{pmatrix} (1\times 0)+(2\times 6) & (1\times 2)+(2\times 3) & (1\times 4)+(2\times 0) & (1\times 9)+(2\times 2) \\ (7\times 0)+(3\times 6) & (7\times 2)+(3\times 3) & (7\times 4)+(3\times 0) & (7\times 9)+(3\times 2) \\ (1\times 0)+(6\times 6) & (1\times 2)+(6\times 3) & (1\times 4)+(6\times 0) & (1\times 9)+(6\times 2) \end{pmatrix}}$$

$$= \underset{(3\times 4)}{\begin{pmatrix} 12 & 8 & 4 & 13 \\ 18 & 23 & 28 & 69 \\ 36 & 20 & 4 & 21 \end{pmatrix}}$$

- The transpose of a matrix, written A' or A^{T} is the matrix obtained by transposing (switching) the rows and columns of a matrix

$$\text{e.g. } A = \begin{pmatrix} 1 & 2 \\ 7 & 3 \\ 1 & 6 \end{pmatrix} \quad A' = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 3 & 6 \end{pmatrix}$$

  If A is $R \times C$, A' will be $C \times R$.

  The rank of a matrix A is given by the maximum number of linearly independent rows (or columns) contained in the matrix. For example, rank

$$\begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix} = 2$$

  since both rows and columns are (linearly) independent of one another, but rank

$$\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = 1$$

  as the second column is not independent of the first (the second column is simply twice the first). A matrix with a rank equal to its dimension, as in the first of these two cases, is known as a *matrix of full rank*. A matrix that is less than of full rank is known as a *short rank matrix*, such a matrix is also termed singular. Three important results concerning the rank of a matrix are: Rank(A) = Rank (A') Rank($A\ B$) $\leq$ min(Rank(A), Rank(B))
- Rank ($A'\ A$) = Rank ($A\ A'$) = Rank (A)
- The inverse of a matrix A, denoted A^{-1}, where defined, is that matrix which, when pre-multiplied or post multiplied by A will result in the identity matrix

$$\text{i.e. } AA^{-1} = A^{-1}A = I$$

  The inverse of a matrix exists only when the matrix is square and non-singular – that is, it is of full rank. The inverse of a 2 × 2 non-singular matrix whose elements are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

will be given by

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The calculation of the inverse of an $N \times N$ matrix for $N > 2$ is more complex and beyond the scope of this text. Properties of the inverse of a matrix include: $I^{-1} = I$ $(A^{-1})^{-1} = A$ $(A')^{-1} = (A^{-1})'$ $(AB)^{-1} = B^{-1}A^{-1}$

- The trace of a square matrix is the sum of the terms on its leading diagonal. For example, the trace of the matrix

$$A = \begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix}$$

written $\mathrm{Tr}(A)$, is $3 + 9 = 12$. Some important properties of the trace of a matrix are: $\mathrm{Tr}(cA) = c\mathrm{Tr}(A)$ $\mathrm{Tr}(A') = \mathrm{Tr}(A)$ $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ $\mathrm{Tr}(I_N) = N$

A6 The eigenvalues of a matrix

Let Π denote a $p \times p$ square matrix and let c denote a $p \times 1$ non-zero vector, and let λ denote a set of scalars. λ is called a *characteristic root* or set of roots of the matrix Π if it is possible to write

$$\underset{p\times p}{\Pi}\,\underset{p\times 1}{c} = \lambda \underset{p\times 1}{c}$$

This equation can also be written as

$$\Pi c = \lambda I_p c$$

where I_p is an identity matrix, and hence

$$(\Pi - \lambda I_p)c = 0$$

Since $c \neq 0$ by definition, then for this system to have a non-zero solution, the matrix $(\Pi - \lambda I_p)$ is required to be singular (i.e. to have zero determinant)

$$|\Pi - \lambda I_p| = 0$$

For example, let Π be the 2×2 matrix

$$\Pi = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$$

Then the characteristic equation is

$$|\Pi - \lambda I_p| = \left| \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda) - 2 = \lambda^2 - 9\lambda + 18$$

This gives the solutions $\lambda = 6$ and $\lambda = 3$. The characteristic roots are also known as *eigenvalues*. The eigenvectors would be the values of c corresponding to the eigenvalues. Some properties of the eigenvalues of any square matrix A are:

- the sum of the eigenvalues is the trace of the matrix
- the product of the eigenvalues is the determinant
- the number of non-zero eigenvalues is the rank.

For a further illustration of the last of these properties, consider the matrix

$$\Pi = \begin{bmatrix} 0.5 & 0.25 \\ 0.7 & 0.35 \end{bmatrix}$$

Its characteristic equation is

$$\left| \begin{bmatrix} 0.5 & 0.25 \\ 0.7 & 0.35 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

which implies that

$$\begin{vmatrix} 0.5 - \lambda & 0.25 \\ 0.7 & 0.35 - \lambda \end{vmatrix} = 0$$

This determinant can also be written $(0.5 - \lambda)(0.35 - \lambda) - (0.7 \times 0.25) = 0$ or

$$0.175 - 0.85\lambda + \lambda^2 - 0.175 = 0$$

or

$$\lambda^2 - 0.85\lambda = 0$$

which can be factorised to $\lambda\ (\lambda - 0.85) = 0$.

The characteristic roots are therefore 0 and 0.85. Since one of these eigenvalues is zero, it is obvious that the matrix Π cannot be of full rank. In fact, this is also obvious from just looking at Π, since the second column is exactly half the first.

Appendix 2

Tables of statistical distributions

Table A2.1 Normal critical values for different values of α

α	0.4	0.25	0.2	0.15	0.1	0.05	0.025	0.01	0.005	0.001
Z_α	.2533	.6745	.8416	1.0364	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

Source: Biometrika Tables for Statisticians (1966), volume 1, 3rd edn. Reprinted with permission of Oxford University Press.

Table A2.2 Critical values of Student's t-distribution for different probability levels, α and degrees of freedom, ν

α	0.4	0.25	0.15	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
ν										
1	0.3249	1.0000	1.9626	3.0777	6.3138	12.7062	31.8205	63.6567	318.3087	636.6189
2	0.2887	0.8165	1.3862	1.8856	2.9200	4.3027	6.9646	9.9248	22.3271	31.5991
3	0.2767	0.7649	1.2498	1.6377	2.3534	3.1824	4.5407	5.8409	10.2145	12.9240
4	0.2707	0.7407	1.1896	1.5332	2.1318	2.7764	3.7469	4.6041	7.1732	8.6103
5	0.2672	0.7267	1.1558	1.4759	2.0150	2.5706	3.3649	4.0321	5.8934	6.8688
6	0.2648	0.7176	1.1342	1.4398	1.9432	2.4469	3.1427	3.7074	5.2076	5.9588
7	0.2632	0.7111	1.1192	1.4149	1.8946	2.3646	2.9980	3.4995	4.7853	5.4079
8	0.2619	0.7064	1.1081	1.3968	1.8595	2.3060	2.8965	3.3554	4.5008	5.0413
9	0.2610	0.7027	1.0997	1.3830	1.8331	2.2622	2.8214	3.2498	4.2968	4.7809
10	0.2602	0.6998	1.0931	1.3722	1.8125	2.2281	2.7638	3.1693	4.1437	4.5869
11	0.2596	0.6974	1.0877	1.3634	1.7959	2.2010	2.7181	3.1058	4.0247	4.4370
12	0.2590	0.6955	1.0832	1.3562	1.7823	2.1788	2.6810	3.0545	3.9296	4.3178
13	0.2586	0.6938	1.0795	1.3502	1.7709	2.1604	2.6503	3.0123	3.8520	4.2208
14	0.2582	0.6924	1.0763	1.3450	1.7613	2.1448	2.6245	2.9768	3.7874	4.1405
15	0.2579	0.6912	1.0735	1.3406	1.7531	2.1314	2.6025	2.9467	3.7328	4.0728
16	0.2576	0.6901	1.0711	1.3368	1.7459	2.1199	2.5835	2.9208	3.6862	4.0150
17	0.2573	0.6892	1.0690	1.3334	1.7396	2.1098	2.5669	2.8982	3.6458	3.9651
18	0.2571	0.6884	1.0672	1.3304	1.7341	2.1009	2.5524	2.8784	3.6105	3.9216
19	0.2569	0.6876	1.0655	1.3277	1.7291	2.0930	2.5395	2.8609	3.5794	3.8834
20	0.2567	0.6870	1.0640	1.3253	1.7247	2.0860	2.5280	2.8453	3.5518	3.8495
21	0.2566	0.6864	1.0627	1.3232	1.7207	2.0796	2.5176	2.8314	3.5272	3.8193
22	0.2564	0.6858	1.0614	1.3212	1.7171	2.0739	2.5083	2.8188	3.5050	3.7921
23	0.2563	0.6853	1.0603	1.3195	1.7139	2.0687	2.4999	2.8073	3.4850	3.7676
24	0.2562	0.6848	1.0593	1.3178	1.7109	2.0639	2.4922	2.7969	3.4668	3.7454
25	0.2561	0.6844	1.0584	1.3163	1.7081	2.0595	2.4851	2.7874	3.4502	3.7251
26	0.2560	0.6840	1.0575	1.3150	1.7056	2.0555	2.4786	2.7787	3.4350	3.7066
27	0.2559	0.6837	1.0567	1.3137	1.7033	2.0518	2.4727	2.7707	3.4210	3.6896
28	0.2558	0.6834	1.0560	1.3125	1.7011	2.0484	2.4671	2.7633	3.4082	3.6739
29	0.2557	0.6830	1.0553	1.3114	1.6991	2.0452	2.4620	2.7564	3.3962	3.6594
30	0.2556	0.6828	1.0547	1.3104	1.6973	2.0423	2.4573	2.7500	3.3852	3.6460
35	0.2553	0.6816	1.0520	1.3062	1.6896	2.0301	2.4377	2.7238	3.3400	3.5911
40	0.2550	0.6807	1.0500	1.3031	1.6839	2.0211	2.4233	2.7045	3.3069	3.5510
45	0.2549	0.6800	1.0485	1.3006	1.6794	2.0141	2.4121	2.6896	3.2815	3.5203
50	0.2547	0.6794	1.0473	1.2987	1.6759	2.0086	2.4033	2.6778	3.2614	3.4960
60	0.2545	0.6786	1.0455	1.2958	1.6706	2.0003	2.3901	2.6603	3.2317	3.4602
70	0.2543	0.6780	1.0442	1.2938	1.6669	1.9944	2.3808	2.6479	3.2108	3.4350
80	0.2542	0.6776	1.0432	1.2922	1.6641	1.9901	2.3739	2.6387	3.1953	3.4163
90	0.2541	0.6772	1.0424	1.2910	1.6620	1.9867	2.3685	2.6316	3.1833	3.4019
100	0.2540	0.6770	1.0418	1.2901	1.6602	1.9840	2.3642	2.6259	3.1737	3.3905
120	0.2539	0.6765	1.0409	1.2886	1.6577	1.9799	2.3578	2.6174	3.1595	3.3735
150	0.2538	0.6761	1.0400	1.2872	1.6551	1.9759	2.3515	2.6090	3.1455	3.3566
200	0.2537	0.6757	1.0391	1.2858	1.6525	1.9719	2.3451	2.6006	3.1315	3.3398
300	0.2536	0.6753	1.0382	1.2844	1.6499	1.9679	2.3388	2.5923	3.1176	3.3233
∞	0.2533	0.6745	1.0364	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

Source: Biometrika Tables for Statisticians (1966), volume 1, 3rd edn. Reprinted with permission of Oxford University Press.

Table A2.3 Upper 5% critical values for *F*-distribution

Degrees of Freedom for denominator ($T-k$)	Degrees of freedom for numerator (m): 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2:33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08.	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

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Table A2.4 Upper 1% critical values for *F*-distribution

Degrees of Freedom for denominator ($T-k$)	Degrees of freedom for numerator (m)																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023	6,056	6,106	6,157	6,209	6,235	6,261	6,287	6,313	6,339	6,366
2	98.5	99.0	99.2	99.3	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.4	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.53	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Source: Biometrika Tables for Statisticians (1966), volume 1, 3rd edn. Reprinted with permission of Oxford University Press.

Table A2.5 Chi-squared critical values for different values of α and degrees of freedom, υ

υ	0.995	0.990	0.975	0.950	0.900	0.750	0.500	0.250	0.100	0.050	0.025	0.010	0.005
1	0.00004	0.00016	0.00098	0.00393	0.01579	0.1015	0.4549	1.323	2.706	3.841	5.024	6.635	7.879
2	0.01003	0.02010	0.05065	0.1026	0.2107	0.5754	1.386	2.773	4.605	5.991	7.378	9.210	10.597
3	0.07172	0.1148	0.2158	0.3518	0.5844	1.213	2.366	4.108	6.251	7.815	9.348	11.345	12.838
4	0.2070	0.2971	0.4844	0.7107	1.064	1.923	3.357	5.385	7.779	9.488	11.143	13.277	14.860
5	0.4117	0.5543	0.8312	1.145	1.610	2.675	4.351	6.626	9.236	11.070	12.833	15.086	16.750
6	0.6757	0.8721	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	16.812	18.548
7	0.9893	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.389	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	6.737	9.342	12.549	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	7.584	10.341	13.701	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	8.438	11.340	14.845	18.54	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.041	9.299	12.340	15.984	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	10.165	13.339	17.117	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	11.036	14.339	18.245	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	11.912	15.338	19.369	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	12.792	16.338	20.489	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	13.675	17.338	21.605	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	14.562	18.338	22.718	27.204	30.143	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	15.452	19.337	23.828	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	16.344	20.337	24.935	29.615	32.670	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	17.240	21.337	26.039	30.813	33.924	36.781	40.289	42.796

23	9.260	10.196	11.688	13.090	14.848	18.137	22.337	27.141	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	19.037	23.337	28.241	33.196	36.415	39.364	42.080	45.558
25	10.520	11.524	13.120	14.611	16.473	19.939	24.337	29.339	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	20.843	25.336	30.434	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	21.749	26.336	31.528	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	22.657	27.336	32.620	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	23.567	28.336	33.711	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	24.478	29.336	34.800	40.256	43.773	46.979	50.892	53.672
35	17.192	18.509	20.569	22.465	24.797	29.054	34.336	40.223	46.059	49.802	53.203	57.342	60.275
40	20.707	22.164	24.433	26.509	29.050	33.660	39.335	45.616	51.805	55.758	59.342	63.691	66.766
45	24.311	25.901	28.366	30.612	33.350	38.291	44.335	50.985	57.505	61.656	65.410	69.957	73.166
50	27.991	29.707	32.357	34.764	37.689	42.942	49.335	56.334	63.167	67.505	71.420	76.154	79.490
55	31.735	33.571	36.398	38.958	42.060	47.611	54.335	61.665	68.796	73.311	77.381	82.292	85.749
60	35.535	37.485	40.482	43.158	46.459	52.294	59.335	66.981	74.397	79.082	83.298	85.379	91.952
70	43.275	45.442	48.758	51.739	55.329	61.698	69.334	77.577	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	71.144	79.334	88.130	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	80.625	89.334	98.650	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	90.133	99.334	109.141	118.498	124.342	129.561	135.807	140.169
120	83.829	86.909	91.568	95.705	100.627	109.224	119.335	130.051	140.228	146.565	152.214	158.963	163.670
150	109.122	112.655	117.980	122.692	126.278	137.987	149.334	161.258	172.577	179.579	185.803	193.219	198.380
200	152.224	156.421	162.724	168.279	174.825	156.175	199.334	213.099	226.018	233.993	241.060	249.455	255.281
250	196.145	200.929	208.095	214.392	221.809	234.580	249.334	264.694	279.947	287.889	295.691	304.948	311.361

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Table A2.6 Lower and upper 1% critical values for Durbin–Watson statistic

	$k'=1$		$k'=2$		$k'=3$		$k'=4$		$k'=5$	
T	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	0.81	1.07	0.70	1.25	0.59	1.46	0.49	1.70	0.39	1.96
16	0.84	1.09	0.74	1.25	0.63	1.44	0.53	1.66	0.44	1.90
17	0.87	1.10	0.77	1.25	0.67	1.43	0.57	1.63	0.48	1.85
18	0.90	1.12	0.80	1.26	0.71	1.42	0.61	1.60	0.52	1.80
19	0.93	1.13	0.83	1.26	0.74	1.41	0.65	1.58	0.56	1.77
20	0.95	1.15	0.86	1.27	0.77	1.41	0.68	1.57	0.60	1.74
21	0.97	1.16	0.89	1.27	0.80	1.41	0.72	1.55	0.63	1.71
22	1.00	1.17	0.91	1.28	0.83	1.40	0.75	1.54	0.66	1.69
23	1.02	1.19	0.94	1.29	0.86	1.40	0.77	1.53	0.70	1.67
24	1.04	1.20	0.96	1.30	0.88	1.41	0.80	1.53	0.72	1.66
25	1.05	1.21	0.98	1.30	0.90	1.41	0.83	1.52	0.75	1.65
26	1.07	1.22	1.00	1.31	0.93	1.41	0.85	1.52	0.78	1.64
27	1.09	1.23	1.02	1.32	0.95	1.41	0.88	1.51	0.81	1.63
28	1.10	1.24	1.04	1.32	0.97	1.41	0.90	1.51	0.83	1.62
29	1.12	1.25	1.05	1.33	0.99	1.42	0.92	1.51	0.85	1.61
30	1.13	1.26	1.07	1.34	1.01	1.42	0.94	1.51	0.88	1.61
31	1.15	1.27	1.08	1.34	1.02	1.42	0.96	1.51	0.90	1.60
32	1.16	1.28	1.10	1.35	1.04	1.43	0.98	1.51	0.92	1.60
33	1.17	1.29	1.11	1.36	1.05	1.43	1.00	1.51	0.94	1.59
34	1.18	1.30	1.13	1.36	1.07	1.43	1.01	1.51	0.95	1.59
35	1.19	1.31	1.14	1.37	1.08	1.44	1.03	1.51	0.97	1.59
36	1.21	1.32	1.15	1.38	1.10	1.44	1.04	1.51	0.99	1.59
37	1.22	1.32	1.16	1.38	1.11	1.45	1.06	1.51	1.00	1.59
38	1.23	1.33	1.18	1.39	1.12	1.45	1.07	1.52	1.02	1.58
39	1.24	1.34	1.19	1.39	1.14	1.45	1.09	1.52	1.03	1.58
40	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
45	1.29	1.38	1.24	1.42	1.20	1.48	1.16	1.53	1.11	1.58
50	1.32	1.40	1.28	1.45	1.24	1.49	1.20	1.54	1.16	1.59
55	1.36	1.43	1.32	1.47	1.28	1.51	1.25	1.55	1.21	1.59
60	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
65	1.41	1.47	1.38	1.50	1.35	1.53	1.31	1.57	1.28	1.61
70	1.43	1.49	1.40	1.52	1.37	1.55	1.34	1.58	1.31	1.61
75	1.45	1.50	1.42	1.53	1.39	1.56	1.37	1.59	1.34	1.62
80	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
85	1.48	1.53	1.46	1.55	1.43	1.58	1.41	1.60	1.39	1.63
90	1.50	1.54	1.47	1.56	1.45	1.59	1.43	1.61	1.41	1.64
95	1.51	1.55	1.49	1.57	1.47	1.60	1.45	1.62	1.42	1.64
100	1.52	1.56	1.50	1.58	1.48	1.60	1.46	1.63	1.44	1.65

Note: T, number of observations; k', number of explanatory variables (excluding a constant term).
Source: Durbin, J. and Watson, G.S. (1951) Testing for serial correlation in least squares regression II *Biometrika*, 38(1–2), 159–177. Reprinted with the permission of Oxford University Press.

Table A2.7 Dickey–Fuller critical values for different significance levels, α

Sample size T	0.01	0.025	0.05	0.10
		τ		
25	−2.66	−2.26	−1.95	−1.60
50	−2.62	−2.25	−1.95	−1.61
100	−2.60	−2.24	−1.95	−1.61
250	−2.58	−2.23	−1.95	−1.62
500	−2.58	−2.23	−1.95	−1.62
∞	−2.58	−2.23	−1.95	−1.62
		τ_μ		
25	−3.75	−3.33	−3.00	−2.63
50	−3.58	−3.22	−2.93	−2.60
100	−3.51	−3.17	−2.89	−2.58
250	−3.46	−3.14	−2.88	−2.57
500	−3.44	−3.13	−2.87	−2.57
∞	−3.43	−3.12	−2.86	−2.57
		τ_τ		
25	−4.38	−3.95	−3.60	−3.24
50	−4.15	−3.80	−3.50	−3.18
100	−4.04	−3.73	−3.45	−3.15
250	−3.99	−3.69	−3.43	−3.13
500	−3.98	−3.68	−3.42	−3.13
∞	−3.96	−3.66	−3.41	−3.12

Source: Fuller (1976). Reprinted with the permission of John Wiley & Sons.

Table A2.8 Critical values for the Engle–Granger cointegration test on regression residuals with no constant in test regression

Number of variables in system	Sample size T	0.01	0.05	0.10
	50	−4.32	−3.67	−3.28
2	100	−4.07	−3.37	−3.03
	200	−4.00	−3.37	−3.02
	50	−4.84	−4.11	−3.73
3	100	−4.45	−3.93	−3.59
	200	−4.35	−3.78	−3.47
	50	−4.94	−4.35	−4.02
4	100	−4.75	−4.22	−3.89
	200	−4.70	−4.18	−3.89
	50	−5.41	−4.76	−4.42
5	100	−5.18	−4.58	−4.26
	200	−5.02	−4.48	−4.18

Source: Engle and Yoo (1987). Reprinted with the permission of Elsevier Science.

Table A2.9 Quantiles of the asymptotic distribution of the Johansen cointegration rank test statistics (constant in cointegrating vectors only)

$p-r$	50%	80%	90%	95%	97.5%	99%	Mean	Var
λ_{max}								
1	3.40	5.91	7.52	9.24	10.80	12.97	4.03	7.07
2	8.27	11.54	13.75	15.67	17.63	20.20	8.86	13.08
3	13.47	17.40	19.77	22.00	24.07	26.81	14.02	19.24
4	18.70	22.95	25.56	28.14	30.32	33.24	19.23	23.83
5	23.78	28.76	31.66	34.40	36.90	39.79	24.48	29.26
6	29.08	34.25	37.45	40.30	43.22	46.82	29.72	34.63
7	34.73	40.13	43.25	46.45	48.99	51.91	35.18	38.35
8	39.70	45.53	48.91	52.00	54.71	57.95	40.35	41.98
9	44.97	50.73	54.35	57.42	60.50	63.71	45.55	44.13
10	50.21	56.52	60.25	63.57	66.24	69.94	50.82	49.28
11	55.70	62.38	66.02	69.74	72.64	76.63	56.33	54.99
λ_{Trace}								
1	3.40	5.91	7.52	9.24	10.80	12.97	4.03	7.07
2	11.25	15.25	17.85	19.96	22.05	24.60	11.91	18.94
3	23.28	28.75	32.00	34.91	37.61	41.07	23.84	37.98
4	38.84	45.65	49.65	53.12	56.06	60.16	39.50	59.42
5	58.46	66.91	71.86	76.07	80.06	84.45	59.16	91.65
6	81.90	91.57	97.18	102.14	106.74	111.01	82.49	126.94
7	109.17	120.35	126.58	131.70	136.49	143.09	109.75	167.91
8	139.83	152.56	159.48	165.58	171.28	177.20	140.57	208.09
9	174.88	198.08	196.37	202.92	208.81	215.74	175.44	257.84
10	212.93	228.08	236.54	244.15	251.30	257.68	213.53	317.24
11	254.84	272.82	282.45	291.40	298.31	307.64	256.15	413.35

Source: Osterwald-Lenum (1992, table 1*). Reprinted with the permission of Blackwell Publishers.

Table A2.10 Quantiles of the asymptotic distribution of the Johansen cointegration rank test statistics (constant, i.e. a drift only in VAR and in cointegrating vector)

$p-r$	50%	80%	90%	95%	97.5%	99%	Mean	Var
λ_{max}								
1	0.44	1.66	2.69	3.76	4.95	6.65	0.99	2.04
2	6.85	10.04	12.07	14.07	16.05	18.63	7.47	12.42
3	12.34	16.20	18.60	20.97	23.09	25.52	12.88	18.67
4	17.66	21.98	24.73	27.07	28.98	32.24	18.26	23.47
5	23.05	27.85	30.90	33.46	35.71	38.77	23.67	28.82
6	28.45	33.67	36.76	39.37	41.86	45.10	29.06	33.57
7	33.83	39.12	42.32	45.28	47.96	51.57	34.37	37.41
8	39.29	45.05	48.33	51.42	54.29	57.69	39.85	42.90
9	44.58	50.55	53.98	57.12	59.33	62.80	45.10	44.93
10	49.66	55.97	59.62	62.81	65.44	69.09	50.29	49.41
11	54.99	61.55	65.38	68.83	72.11	75.95	55.63	54.92
λ_{Trace}								
1	0.44	1.66	2.69	3.76	4.95	6.65	0.99	2.04
2	7.55	11.07	13.33	15.41	17.52	20.04	8.23	14.38
3	18.70	23.64	26.79	29.68	32.56	35.65	19.32	32.43
4	33.60	40.15	43.95	47.21	50.35	54.46	34.24	52.75
5	52.30	60.29	64.84	68.52	71.80	76.07	52.95	79.25
6	75.26	84.57	89.48	94.15	98.33	103.18	75.74	114.65
7	101.22	112.30	118.50	124.24	128.45	133.57	101.91	158.78
8	131.62	143.97	150.53	156.00	161.32	168.36	132.09	201.82
9	165.11	178.90	186.39	192.89	198.82	204.95	165.90	246.45
10	202.58	217.81	225.85	233.13	239.46	247.18	203.39	300.80
11	243.90	260.82	269.96	277.71	284.87	293.44	244.66	379.56

Source: Osterwald-Lenum (1992, table 1). Reprinted with the permission of Blackwell Publishers.

Table A2.11 Quantiles of the asymptotic distribution of the Johansen cointegration rank test statistics (constant in cointegrating vector and VAR, trend in cointegrating vector)

$p - r$	50%	80%	90%	95%	97.5%	99%	Mean	Var
λ_{max}								
1	5.55	8.65	10.49	12.25	14.21	16.26	6.22	10.11
2	10.90	14.70	16.85	18.96	21.14	23.65	11.51	16.38
3	16.24	20.45	23.11	25.54	27.68	30.34	16.82	22.01
4	21.50	26.30	29.12	31.46	33.60	36.65	22.08	27.74
5	26.72	31.72	34.75	37.52	40.01	42.36	27.32	31.36
6	32.01	37.50	40.91	43.97	46.84	49.51	32.68	37.91
7	37.57	43.11	46.32	49.42	51.94	54.71	38.06	39.74
8	42.72	48.56	52.16	55.50	58.08	62.46	43.34	44.83
9	48.17	54.34	57.87	61.29	64.12	67.88	48.74	49.20
10	53.21	59.49	63.18	66.23	69.56	73.73	53.74	52.64
11	58.54	64.97	69.26	72.72	75.72	79.23	59.15	56.97
λ_{Trace}								
1	5.55	8.65	10.49	12.25	14.21	16.26	6.22	10.11
2	15.59	20.19	22.76	25.32	27.75	30.45	16.20	24.90
3	29.53	35.56	39.06	42.44	45.42	48.45	30.15	45.68
4	47.17	54.80	59.14	62.99	66.25	70.05	47.79	74.48
5	68.64	77.83	83.20	87.31	91.06	96.58	69.35	106.56
6	94.05	104.73	110.42	114.90	119.29	124.75	94.67	143.33
7	122.87	134.57	141.01	146.76	152.52	158.49	123.51	182.85
8	155.40	169.10	176.67	182.82	187.91	196.08	156.41	234.11
9	192.37	207.25	215.17	222.21	228.05	234.41	193.03	288.30
10	231.59	247.91	256.72	263.42	270.33	279.07	232.25	345.23
11	276.34	294.12	303.13	310.81	318.02	327.45	276.88	416.98

Source: Osterwald-Lenum (1992, table 2*). Reprinted with the permission of Blackwell Publishers.

Appendix 3

Sources of data used in this book

I am grateful to the following organisations, who all kindly agreed to allow their data to be used as examples in this book and for it to be copied onto the book's web site: Bureau of Labor Statistics, Federal Reserve Board, Federal Reserve Bank of St. Louis, Nationwide, Oanda, and Yahoo! Finance. The following table gives details of the data used and of the provider's web site.

Provider	Data	Web
Bureau of Labor Statistics	CPI	http://www.bls.gov
Federal Reserve Board	US T-bill yields, money supply, industrial production, consumer credit	http://www.federalreserve.gov
Federal Reserve Bank of St. Louis	average AAA & BAA corporate bond yields	http://research.stlouisfed.org/fred2
Nationwide	UK average house prices	http://www.nationwide.co.uk
Oanda	euro–dollar, pound–dollar & yen–dollar exchange rates	http://www.oanda.com/convert/fxhistory
Yahoo! Finance	S&P500 and various US stock and futures prices	http://finance.yahoo.com